Sep. 13.

Question 2023 Midterm 1

(a) $\max\ x_1^{\alpha} x_2^{1-\alpha}$

s.t. $p_1x_1 + p_2x_2 \leq M$

$$\mathcal{L} = x_1^{\alpha} x_2^{1-\alpha} - \lambda(p_1x_1 + p_2x_2 - M)$$

Here, since $MU_1 > 0$. $MU_2 > 0$ both goods are normal. the constraint must be binding.

FOC. $\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 = 0$

$$\frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 = 0$$

$$\lambda(p_1x_1 + p_2x_2 - M) = 0. \quad \lambda \geq 0, \quad \Rightarrow \quad p_1x_1 + p_2x_2 = M$$

$$\frac{\alpha}{(1-\alpha)} \frac{x_2}{x_1} = \frac{p_1}{p_2} \iff \alpha p_2 x_2 = (1-\alpha) p_1 x_1$$

substitute this back to the budget constraint.

$$x_1^m(p, M) = \frac{\alpha M}{p_1} \qquad x_2^m(p, M) = \frac{(1-\alpha)M}{p_2}$$

(b) Indirect utility function.

$$v(p_1, p_2, M) = (x_1^m)^{\alpha} (x_2^m)^{1-\alpha}$$

$$= \left(\frac{\alpha M}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)M}{p_2}\right)^{1-\alpha}$$

$$= \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} M$$

(c). Verify the Roy's identity:

$$x_1^m = -\frac{\frac{\partial V}{\partial p_1}}{\frac{\partial V}{\partial M}} = -\frac{(-\alpha)\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}\frac{M}{p_1}}{\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}} = \frac{\alpha M}{p_1} \qquad \text{verified.}$$

x_2^m could be verified in the same way.

Question
JR Example
2.1

$V(p_1, p_2, M) = M(p_1^r + p_2^r)^{-\frac{1}{r}}$. Recover the corresponding direct utility

Step 1: let $M=1$. $V(p_1, p_2, 1) = (p_1^r + p_2^r)^{-\frac{1}{r}}$

Step 2: $\min V(p_1, p_2)$ s.t. $p_1x_1 + p_2x_2 \leq 1$

$$\mathcal{L} = -(p_1^r + p_2^r)^{-\frac{1}{r}} - \lambda(p_1x_1 + p_2x_2 - 1)$$

FOC:

$$[p_1]: -(-\tfrac{1}{r})(p_1^r + p_2^r)^{-\frac{1}{r}-1} \cdot r\, p_1^{r-1} - x_1\lambda = 0$$

$$[p_2]: -(-\tfrac{1}{r})(p_1^r + p_2^r)^{-\frac{1}{r}-1}\, r\, p_2^{r-1} - x_2\lambda = 0$$

$$\lambda(p_1x_1 + p_2x_2 - 1) = 0$$

$$\Rightarrow \left(\frac{p_1}{p_2}\right)^{r-1} = \frac{x_1}{x_2} \qquad \frac{p_1}{p_2} = \left(\frac{x_1}{x_2}\right)^{\frac{1}{r-1}} \Rightarrow p_1 = \left(\frac{x_1}{x_2}\right)^{\frac{1}{r-1}} p_2$$

Substitute into $p_1x_1 + p_2x_2 = 1$

$$\left(\frac{x_1}{x_2}\right)^{\frac{1}{r-1}} p_2 x_1 + p_2 x_2 = 1$$

$$\Rightarrow p_2\left[\left(\frac{x_1}{x_2}\right)^{\frac{1}{r-1}} \cdot x_1 + x_2\right] = 1$$

$$\Rightarrow p_2^* = \frac{x_2^{\frac{1}{r-1}}}{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}}$$

$$\text{and } p_1^* = \frac{x_1^{\frac{1}{r-1}}}{x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}}$$

Step 3: substitute p_1^*, p_2^* back

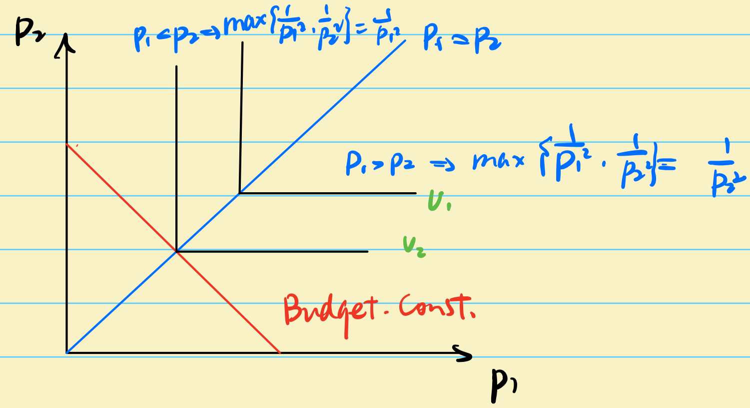
$$u(x_1, x_2) = V(p_1^*, p_2^*) = \left(x_1^{\frac{r}{r-1}} + x_2^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}$$

Question $V(p_1, p_2, m) = \max\left\{\frac{M^2}{p_1^2}, \frac{m^2}{p_2^2}\right\}$

Step 1: let $M=1$, $V(p_1, p_2, 1) = \max\left\{\frac{1}{p_1^2}, \frac{1}{p_2^2}\right\}$

Step 2: $\min\limits_{p_1, p_2} \max\left\{\frac{1}{p_1^2}, \frac{1}{p_2^2}\right\}$

s.t. $p_1 x_1 + p_2 x_2 \leq 1$



therefore, the minimum is achieved when $p_1 = p_2$

thus $p_1^* = p_2^* = \frac{1}{x_1 + x_2}$

Step 3: Substitute p_1^*, p_2^* back

$u(x_1, x_2) = (x_1 + x_2)^2$

Note that in the graph above, the indifference curve closer to the lower and left represents a higher utility, BUT, since our goal is a minimization problem, we will shift the indifference curve as far as possible toward the upper right within the range of budget constraint.

MWG 3.C.6. $u(x) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$, (assume $\alpha_1 + \alpha_2 = 1$)

(a) $\rho \to 1 \Rightarrow u(x) = \alpha_1 x_1 + \alpha_2 x_2$

(b) $\rho \to 0$,

$$u(x) = \exp\left(\frac{1}{\rho}\log(\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)\right)$$

Taylor expansion at x_0

$f(x) = f(x_0) + f'(x_0)(x - x_0) + O(x^2)$

then consider the first order Taylor expansion centered at $\rho = 0$ for the term inside logarithm. with respect to ρ.

$$\alpha_1 x_1^\rho + \alpha_2 x_2^\rho = \alpha_1 x_1^0 + \alpha_2 x_2^0 + \alpha_1 \rho x_1^\rho \ln x_1 + \alpha_2 \rho x_2^\rho \ln x_2 + O(\rho^2)$$

$$= \underbrace{\alpha_1 + \alpha_2}_{1} + \rho\alpha_1 \ln x_1 + \rho\alpha_2 \ln x_2 + O(\rho^2)$$

$$= 1 + \rho \ln(x_1^{\alpha_1} x_2^{\alpha_2}) + O(\rho^2)$$

Big O notation $O(\rho^2)$ remaining approximation that goes to 0 as $\rho \to 0$

plug in back to the utility function

$$u(x) = \exp\left[\frac{1}{\rho}\log(1 + \rho\ln(x_1^{\alpha_1} x_2^{\alpha_2}) + O(\rho^2))\right]$$

$$u(x) = \left[1 + \rho\ln(x_1^{\alpha_1} x_2^{\alpha_2}) + O(\rho^2)\right]^{\frac{1}{\rho}}$$

When $\rho \to 0$, $u(x) = e^{\ln x_1^{\alpha_1} x_2^{\alpha_2}}$

$$\Rightarrow u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$$

$$\lim_{x \to 0} (1 + Ax)^{\frac{1}{x}}$$

$$= \lim_{x \to 0} \exp\left[\frac{\ln(1 + Ax)}{x}\right] = e^{A}.$$

(c). When $\rho \to -\infty$

Discuss by case, when $x_1 > x_2$

$$\lim_{\rho \to -\infty} (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{\frac{1}{\rho}}$$

$$= \lim_{\rho \to -\infty} x_2 \left(\alpha_1 \left(\frac{x_1}{x_2}\right)^{\rho} + \alpha_2 \right)^{\frac{1}{\rho}}$$

let $r = \frac{x_1}{x_2} > 1$

$r > 1 \Rightarrow \ln r > 0$

$$\lim_{\rho \to -\infty} x_2 \left(\alpha_1 r^{\rho} + \alpha_2 \right)^{\frac{1}{\rho}}$$

$$= \lim_{\rho \to -\infty} \exp\left[\frac{\ln(\alpha_1 r^{\rho} + \alpha_2)}{\rho} + \ln x_2 \right]$$

$$= \exp[\, 0 + \ln x_2 \,]$$

$$= x_2$$

Similarly, we can prove when $x_1 < x_2$,

$$\lim_{\rho \to -\infty} \left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho} \right)^{\frac{1}{\rho}} = x_1$$

Therefore, $u(x) = \min\{x_1, x_2\}$

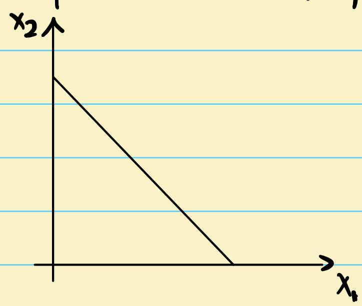
Sometimes we let $\rho = \frac{\sigma - 1}{\sigma}$ and $\sigma = \frac{1}{1-\rho}$,

where σ has a meaning of the elasticity of substitution.

Hence, the CES utility function is given by

$$u(x) = \left(\alpha_1 x_1^{\frac{\sigma-1}{\sigma}} + \alpha_2 x_2^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$$

when $\rho \to 1$, $\sigma \to \infty$, perfect substitute



when $\rho \to 0$ $\sigma = 1$ $\Rightarrow$ Cobb-Douglas

when $\rho \to -\infty$ $\sigma = 0$ $\Rightarrow$ perfect complement